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POSTBUCKLING BEHAVIOR OF AN IDEAL BAR ON AN ELASTIC FOUNDATION

N. S. Astapov and V. M. Kornev

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The differential equation of equilibrium of a bent bar axis [1-3] or the integral expression of the system's potential energy [1, 4-7] are conventionally used as the starting expression in analyzing the stability of a bar on elastic foundation.

Equal values of the critical loads for the buckling of the system are obtained in both cases. With the advent of the catastrophe theory, these results were elucidated from a new, more common viewpoint providing a clear description of the influence of initial imperfections on the behavior of the system.

Nevertheless, postbuckling behavior of the bar–foundation system has not been sufficiently studied. In the present paper the modes of buckling and postbuckling behavior are studied by the perturbation theory method within the framework of three mathematical models, two of which are classical. It appeared that all three models provide dissimilar description of the postbuckling behavior of the system. A distinctive feature of the problems under consideration is the fact that several possible forms of bar buckling correspond to certain values of rigidity of an elastic foundation, i.e., the appropriate eigenfunction and eigennumber problems have multiple eigenvalues.

1. Statement of the Problem. Consider a hinge-supported bar of length L lying on elastic foundation and loaded by axial compression P whose value and direction remain invariant upon deformation of the bar (Fig. 1). The length L of the bar axis is assumed to be invariable. Denote the distance between the bar ends by l. Assume that the bar axis may bend in the plane (x, y) only. Let us study the buckling modes and postbuckling behavior of the bar–foundation system with the use of different models (Figs. 1 and 2) describing the behavior of the system.

2. Classical Model of an Elastic Foundation. Suppose that, in bending, the reaction forces of an elastic foundation at every point of the bar are invariably directed upright to the Ox axis (Fig. 1) and proportional to the bar deflection. In this case the expression for the total potential energy of the bar is written [5]

$$U = \frac{1}{2} EI \int_0^L \varkappa^2 ds - P(L - l) + \frac{1}{2} c \int_0^L w^2 ds, \tag{2.1}$$

where EI is the bending stiffness, $\varkappa$ is the curvature of the bar axis, c is the stiffness coefficient of the foundation, and s is the length of the arc of the bar axis. The function w(s) ($0 \le s \le L$) completely determines the strained state of the bar and should satisfy the geometrical boundary conditions of the problem:

$$w(0) = w(L) = w_{ss}(0) = w_{ss}(L) = 0.$$

Let us express the curvature $\varkappa$ and distance, l in terms of the function w(s) and substitute it into Eq. (2.1). To accuracy up to fourth-order terms involving the function w(s) and its derivatives, we obtain

$$U = \frac{1}{2} EI \int_0^L w_{ss}^2 (1 + w_s^2) ds - P \int_0^L \left(\frac{1}{2} w_s^2 + \frac{1}{8} w_s^4 \right) ds + \frac{1}{2} c \int_0^L w^2 ds. \tag{2.2}$$

The Euler equation for this functional may be written as

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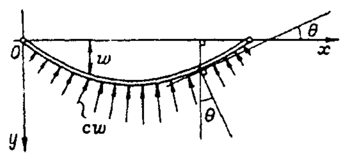

Fig. 1

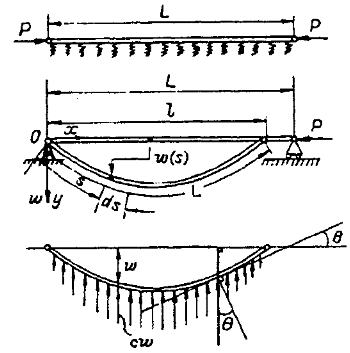

Fig. 2

$$EIw_{ssss} + EI(w_{ss}^2 + 4w_s w_{sss})w_{ss} + P(1 + w_s^2/2)w_{ss} + cw(1 - w_s^2) = 0. \tag{2.3}$$

We introduce a new variable $z = \pi s/L$ and function W such that $w = \alpha W$, where α is of the same order of smallness as the deflection amplitude. Denote $r = c(L/\pi)^4/EI$, and by ε, the small parameter: $\varepsilon = \pi^2(\alpha/L)^2$. With this notation Eq. (2.3) and the boundary conditions take the form

$$(A_0 + \varepsilon A_1)W - \Lambda(B_0 + \varepsilon B_1)W = 0, W(0) = W(\pi) = W_{zz}(0) = W_{zz}(\pi) = 0. \tag{2.4}$$

Here $\Lambda = P(L/\pi)^2 EI$; A_0, B_0 and A_1, B_1 are linear and nonlinear operators,

$$A_0 = (\ldots)_{zzzz} + r(\ldots), \qquad B_0 = -(\ldots)_{zz},$$
$$A_1 = (\ldots)_{zz}^3 + 4(\ldots)_z(\ldots)_{zz}(\ldots)_{zzz} - r(\ldots)(\ldots)_z^2, \quad B_1 = -(\ldots)_z^2(\ldots)_{zz}/2.$$

The trivial solution $W \equiv 0$ of the problem (2.4) relates to unbuckled equilibrium of the system for all Λ, the magnitude of potential energy (2.2) being equal to zero in this case. Further, we will be concerned only with curved equilibrium states. The eigenfunction and eigennumber problem (2.4) for $\varepsilon = 0$ is called unperturbed. We obtain the eigenfunction

$$W_n^{(0)} = \gamma_n \sin nz$$

and the eigennumbers of the unperturbed (linearized) problem

$$\Lambda_n^{(0)} = n^2 + r/n^2,$$

the eigennumbers with numbers n and n + 1 being multiples, i.e., $\Lambda_n^{(0)} = \Lambda_{n+1}^{(0)}$, if $r = n^2(n+1)^2$. Consider the two cases separately.

A. Prime Eigenvalue, i.e., $r \neq n^2(n+1)^2$. Let us calculate the eigennumbers and eigenfunctions of the problem (2.4) using the perturbation theory method [8, 9]. For this purpose, we represent the eigenvalues W_n and eigennumbers Λ_n of the perturbed problem as asymptotic series with respect to ε:

$$W_n = W_n^{(0)} + \sum_{k=1}^{\infty} \varepsilon^k W_n^{(k)}, \; \Lambda_n = \Lambda_n^{(0)} + \sum_{k=1}^{\infty} \varepsilon^k \Lambda_n^{(k)} \tag{2.5}$$

where $W_n^{(0)}$ and $\Lambda_n^{(0)}$ are the eigenvalues and numbers of the unperturbed problem. The normalization condition for the unperturbed problem are preset by the relationship $(B_0 W_i^{(0)}, W_j^{(0)}) = \sigma_{ij}$, ($\sigma_{ij}$ are the Kronecker symbols) and, for the perturbed problem, also by the relationship $(W_i^{(0)}, W_i^{(1)}) = 0$ [9]. We substitute the asymptotic expansions (2.5) into the equation and boundary conditions of the problem (2.4) and equate to zero the coefficients of the powers of ε. Using the normalization conditions, we find the expansions Λ_n and W_n to accuracy up to first power terms in ε. We can show the coincidence of the accuracy of ε^2 of the thus computed eigenvalues with the exact values for a bar without elastic foundation, i.e., when $r = 0$. In our case, substituting the expansion (2.5) in Eqs. (2.4) and equating to zero the coefficient of ε, we obtain

$$A_0W_n^{(1)} + A_1W_n^{(0)} - \Lambda_n^{(0)}(B_0W_n^{(1)} + B_1W_n^{(0)}) - \Lambda_n^{(1)}B_0W_n^{(0)} = 0. \tag{2.6}$$

Substituting the expansion of the function $W_n^{(1)}$ into a series in Eq. (2.6),

$$W_n^{(1)} = \sum_{j=1}^{\infty} \alpha_{nj} W_j^{(0)}$$

with respect to the eigenfunctions $W_n^{(0)}$ of the unperturbed equation and using the normalization conditions, we obtain finally

$$\Lambda_n^{(1)} = (A_1W_n^{(0)}, W_n^{(0)}) - \Lambda_n^{(0)}(B_1W_n^{(0)}, W_n^{(0)}),\ \alpha_{nn} = 0,$$
$$\alpha_{nj} = [\Lambda_n^{(0)}(B_1W_n^{(0)}, W_j^{(0)}) - (A_1W_n^{(0)}, W_j^{(0)})]/(\Lambda_j^{(0)} - \Lambda_n^{(0)}),\ j \neq n.$$

Having calculated these quantities, we find that the eigenfunctions and eigennumbers of the perturbed problem (2.4), with accuracy up to ε, have the form

$$W_n(z) = \gamma_n \sin nz - 9\varepsilon(3n^4 - r)\gamma_{3n}\sin 3nz/(9n^4 - r)/32\pi,$$
$$\Lambda_n = n^2 + r/n^2 + \varepsilon\gamma_n^2(n^4 - 3r)/8,$$

where $\gamma_n^2 n^2 = 2/\pi$ from the normalization conditions. The expression for Λ_n may be written as

$$P = EI(\pi/L)^2[n^2 + r/n^2 + \pi^2\gamma_n^2(n^4 - 3r)(\alpha/L)^2/8]. \tag{2.7}$$

Analyzing (2.7), we observe that the postbuckling behavior of the system is stable with respect to the mode $w_n(s) = \alpha W_n(z(s))$ when $n^4 > 3r$, and unstable when $n^4 < 3r$. If $n^4 = 3r$, the postbuckling behavior of the system is neutral to accuracy up to ε.

The dependence $\Lambda_n^{(0)}$ on r for n = 1, 2, 3, 4 [2, 4, 5] is shown in Fig. 3. At the bottom the influence of maximum deflection on load is shown schematically for each of the mode of w_n (n = 1, 2, 3, 4) in the subdomains into which the range of r is split by the abscjssas of the points of intersection of the straight lines and the values $r = n^4/3$ (n = 1, 2, 3, 4). The points of the straight lines $\Lambda_n^{(0)}$ with abscissas equal to $n^4/3$ are denoted by the symbol H_n^T. The initial model chosen for describing the behavior of the system determines the arrangement of the points H_n at appropriate straight lines, for example, Eq. (2.3). It should be noted that only one branch of the load–deflection dependence is realized. We describe in more detail the postbuckling behavior of the system. The value α corresponding to the mode $w_n = \alpha W_n$ is calculated for a specific value of the load P from Eq. (2.7) and is henceforth denoted by α_n. Since the deflection amplitude of the w_n mode is approximately equal to $\alpha_n\gamma_n$, it makes sense to consider only such values of α_n for which $|\alpha_n\gamma_n| < L/(2n)$. A part ($r < 2n^4/3$) of the plot of the function $\Lambda_n = \Lambda_n(\alpha_n\gamma_n/L, r)$ describing stable ($r < n^4/3$) and unstable ($n^4/3 < r < 2n^4/3$) postbuckling behavior of the ideal system that assumes the w_n mode is presented in Fig. 4. When r = 0, the classical problem of buckling of a bar without foundation is obtained, the postbuckling behavior of this system being always stable. the line of intersection of the surface Λ_n and the plane r = const is a parabola whose branches are directed upward when $r < n^4/3$, and downward when $r > n^4/3$, and downward when $r > n^4/3$. The cross section of the surface Λ_n and the plane $\alpha_n\gamma_n/L$ = const is a straight line.

B. Multiple Eigenvalue, i.e., $r = n^2(n + 1)^2$. Denote the points of intersection of the straight lines $\Lambda_n^{(0)}$ and $\Lambda_{n+1}^{(0)}$ by the symbol $K_{n,\,n+1}$ (Fig. 3) and consider the postbuckling behavior of the system in the neighborhood of these points, i.e, for small deflections. In this case several eigenfunctions correspond to one eigenvalue, and the appropriate formulas take a different form. Unlike Eq. (2.5) the eigenfunctions and eigennumbers of the perturbed problem (2.4) will be sought as follows:

$$W_m = f_m^{(0)} + \sum_{k=1}^{\infty} \varepsilon^k W_m^{(k)},\ \Lambda_m = \Lambda_m^{(0)} + \sum_{k=1}^{\infty} \varepsilon^k \Lambda_m^{(k)},$$
$$f_m^{(0)} = \rho_{mn}W_n^{(0)} + \rho_{m,n+1}W_{n+1}^{(0)},\ m = n, n+1. \tag{2.8}$$

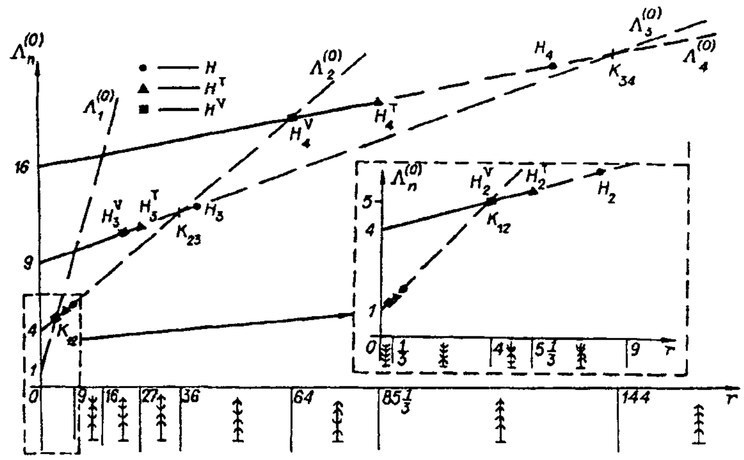

Fig. 3

Substituting the expansions (2.8) in the Eq. (2.4), equating to zero the coefficient of ε, and obtaining scalar product of the result by $W_m^{(0)}$, we obtain a system of two equations for determining $\Lambda_n^{(1)}$ (and analogously for $\Lambda_{n+1}^{(1)}$):

$$\Lambda_m^{(0)}(B_1 f_m^{(0)}, W_m^{(0)}) + \Lambda_m^{(1)}(B_0 f_m^{(0)}, W_m^{(0)}) - (A_1 f_m^{(0)}, W_m^{(0)}) = 0, \; m = n, n + 1, \tag{2.9}$$

as well as the relations for calculating the coefficient α_{mj}:

$$\alpha_{mj} = [\Lambda_m^{(0)}(B_1 f_m^{(0)}, W_j^{(0)}) - (A_1 f_m^{(0)}, W_j^{(0)})]/(\Lambda_j^{(0)} - \Lambda_m^{(0)}), \; j \neq m = n, n + 1.$$

From the system of equations (2.9) we express $\rho_{m,\, n+1}$ and $\Lambda_m^{(1)}$ in terms of ρ_{mn} for $\rho_{m,\, n+1} \neq 0$, $\rho_{mn} \neq 0$:

$$\rho^2_{m,n+1} = [3n^2 - (n + 1)^2]/[3(n+1)^2 - n^2]\rho^2_{mn} = R(n)\rho^2_{mn}; \tag{2.10}$$

$$\Lambda_m^{(1)} = \frac{1}{4\pi}\frac{-7n^4 + 2n^2(n+1)^2 - 7(n+1)^4}{3(n+1)^2 - n^2}\rho^2_{mn} = \frac{1}{4\pi}D(n)\rho^2_{mn}. \tag{2.11}$$

Note that the coefficient R(n) in Eq. (2.10) is greater than zero for all $n \geq 2$. Consequently, apart from the modes w_n and w_{n+1}, for the foundation stiffness $r = n^2 (n + 1)^2$ the system can take the following forms, when $n \geq 2$:

$$w_m^*(s) = \alpha\{\rho_{mn}\gamma_n \sin(n\pi s/L) + \rho_{m,n+1}\gamma_{n+1}\sin((n + 1)\pi s/L) + O(\varepsilon)\}, \; m = n, n + 1, \tag{2.12}$$

where the coefficients ρ_{mj} are connected by the relationship (2.10), and the symbol $O(\varepsilon)$ denotes terms of the first degree and higher with respect to ε. The appropriate eigennumbers with accuracy p to $\varepsilon = \pi^2(\alpha/L)^2$ will be of the form

$$\Lambda_m^* = n^2 + (n + 1)^2 + \pi D(n)\rho^2_{mn}(\alpha/L)^2/4, \; m = n, n + 1. \tag{2.13}$$

Postbuckling behavior of the system in this case is unstable, because $D(n) < 0$ for every $n > 0$. As a result, we have the following: At the point K_{12}, i.e., for $r = 4$, postbuckling behavior is stable with respect to the w_2 mode and unstable with respect to the w_1 mode, and there is no other curved mode at this point. At all other points $K_{n,\, n+1}$ $(n \geq 2)$ the postbuckling behavior of the system is unstable with respect to all of the possible modes, which can be easily checked.

Let us consider the behavior of the system at the points $K_{n,\, n+1}$ $(n \geq 2)$. First, we compare the maximum deflections of the modes w_m and w_m^* $(m = n, n + 1)$ for a fixed value of the load. Using Eq. (2.7), we obtain

$$(\alpha_{n+1}\gamma_{n+1})^2 = [3r - n^4]/[3r - (n + 1)^4](\alpha_n\gamma_n)^2 > (\alpha_n\gamma_n)^2$$

for every $n \geq 2$, i.e., the maximum deflection of the mode w_{n+1} is greater than the w_n deflection. We estimate the value of the maximum deflection the w_n^* mode to the second order with respect to α^2. Taking into account that $R(n) < 1$ for $n \geq 2$, we obtain

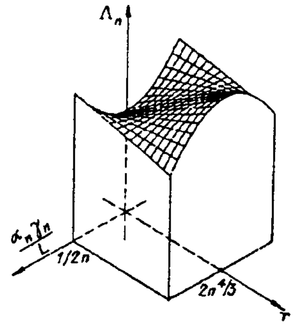

Fig. 4

$$\max_{0<z<\pi}|w_n^*| \approx \max_{0<z<\pi}\alpha|\rho_{nn}\gamma_n\sin nz + \rho_{n,n+1}\gamma_{n+1}\sin(n+1)z| \geq$$
$$\alpha\gamma_{n+1}|\rho_{nn}|\ |(n+1)/n + \sqrt{R(n)}\cos\pi/2n| > \alpha\gamma_{n+1}|\rho_{nn}|(1 + \sqrt{R(n)}).$$

Moroever, it is evident that

$$\max_{0<z<\pi}|w_n^*| < \alpha\gamma_{n+1}|\rho_{nn}|((n+1)/n + \sqrt{R(n)}).$$

We compare the maximum deflections of the modes w_{n+1} and w_n^*. Using Eqs. (2.7) and (2.13), we express $\alpha_{\rho nn}$ in terms of α_{n+1} as follows:

$$(\alpha\rho_{nn})^2 = [(n+1)^2 - 3n^2]/D(n)\alpha_{n+1}^2 = D_1(n)\alpha_{n+1}^2.$$

After calculations, we obtain, for $n \leq 5$,

$$\max_{0<z<\pi} w_n^{*2} < D_1(n)\ ((n+1)/n + \sqrt{R(n)})^2(\alpha_{n+1}\gamma_{n+1})^2 < (\alpha_{n+1}\gamma_{n+1})^2,$$

i.e,, the deflection of the w_n^* mode is less than the maximum deflection of the w_{n+1} mode. However, since

$$D_1(n) < D_1(n+1) < \lim_{m\to\infty} D_1(m) = 1/3,\ R(n) < R(n+1) < \lim_{m\to\infty} R(m) = 1,$$

It can be easily shown that, for all $n \geq 0$, the maximum deflection of the w_n^* mode is larger than the w_{n+1} deflection. It can be proved in the same manner that the maximum w_n^* deflection is always larger than the w_n deflection. Finally, we show that the maximum deflections of the w_n^* and w_{n+1}^* modes coincide. Using Eq. (2.13), we note that $(\alpha\rho_{nn})^2 = (\alpha\rho_{n+1,n})^2$ for one and the same magnitude of the load P. Since

$$\max_{0<z<\pi}|a\sin nz + b\sin(n+1)z| = \max_{0<z<\pi}|a\sin nz - b\sin(n+1)z|,$$

the maximum deflections of the w_n^* and w_{n+1}^* modes coincide. Figure 5 presents two markedly different cases of the mutual arrangement of the parabolas, demonstrating the dependence of the maximum deflection on the load for all of he possible curved buckling modes. Now let us classify the different buckling modes for any fixed value of the load P by means of their corresponding values of potential energy [10]. From the expression (2.2) and the analytical representation of the buckling modes, it is evident that to calculate the potential energy to accuracy up to α^4 it suffices to take into account terms containing α to the first power in expressing each of the possible buckling modes. For the w_n we have the potential energy

$$U_n = L(\pi/L)^6 EI\alpha_n^4\gamma_n^4 n^2[-n^4 + 3r]/64 \tag{2.14}$$

and for the w_{n+1} mode

$$U_{n+1} = L(\pi/L)^6 EI\alpha_{n+1}^4\gamma_{n+1}^4(n+1)^2\ [\ -(n+1)^4 + 3r]/64.$$

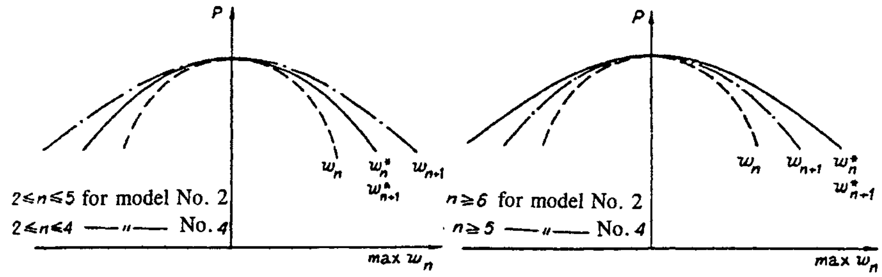

Fig. 5

Since for $r = n^2(n+1)^2$ $(n \geq 2)$ and fixed load P we have $(\alpha_{n+1}\gamma_{n+1})^2 = (n^4 - 3r)/[(n+1)^4 - 3r](\alpha_n\gamma_n)^2$, it follows that $U_{n+1} > U_n$ for any $n \geq 2$. Expressing $\rho_{n,\,n+1}$ in terms of ρ_{mn} (2.10), and $\alpha\rho_{nn}$ in terms of α_n, we find the potential energy for the w_n^* mode:

$$U_n^* = L(\pi/L)^6 EI\alpha_n^4\gamma_n^4 n^4[n^2 + (n+1)^2]\,[n^2 - 3(n+1)^2]^2/\\ [7n^4 - 2n^2(n+1)^2 + 7(n+1)^4]/32.$$

Hence, we deduce that $U_n > U_n^* > 0$ for any $n \geq 2$. Moreover, $U_n^* = U_{n+1}^*$ because $\rho_{n+1,n}^2 = \rho_{nn}^2$ and the equality (2.10) holds, and the coefficients $\rho_{n+1,\,m}$ included in the expression for the potential energy U_{n+1}^* are everywhere to the even power. Thus, we find that for fixed values of the stiffness coefficients of the foundation $r = n^2(n+1)^2$ $(n \geq 2)$ and load P, the energies of the branches of the solution are arranged as follows: $U_{n+1} > U_n > U_n^* = U_{n+1}^*$. Assuming that the system selects the state with the least energy, the appropriate buckling mode is of the form w_n^* or w_{n+1}^*. However, the potential energy (2.2) of the system having the uncurved mode $w \equiv 0$ is equal to zero, and, moreover, $U_n^* = U_{n+1}^* > 0$. Thus, the rectilinear equilibrium state $w \equiv 0$ is the most probable in this model. Note that the expression (2.14) for the potential energy of the system assumes the mode w_n is in the form of the germ of a cusp catastrophe, when $r > n^4/3$; in the form of dual cusp, when $r < n^4/3$; and in the form of the germ of a more complicated catastrophe than the cusp, when $r = n^4/3$. To put this in another way, in this model a change in the type of catastrophe is observed at the points H_n^T, i.e., the requirement of structural stability of the family of potential functions is violated [11, 12]. Precisely, when $r = n^4/3$, the postbuckling behavior of the system with respect to the w_n mode becomes unstable (2.7), and the most probable among the unstable branches is rectilinear equilibrium.

3. Nonclassical Model. Let us consider a somewhat different model in which the total potential energy is expressed as follows:

$$U = \frac{1}{2}EI\int_0^L \varkappa^2 ds - P(L - l) + c\int_0^L\int_0^w w(1 - w_s^2)^{1/2} dw\,ds. \tag{3.1}$$

Formula (3.1) differs from Eq. (2.1) only in the multiplier $(1 - w_s^2)^{1/2}$ which at any point of the bar has the form

$$(1 - w_s^2)^{1/2} = \cos\theta(s),$$

where $\theta(s)$ is the angle between the Ox axis and the tangent to the bar axis at this point. Alternatively, the angle $\theta(s)$ is equal to that between the normals to the bar axis and to the Ox axis (Fig. 2). Thus, this model differs physically from the preceding one (Fig. 1) in setting the reaction forces of an elastic foundation. Here (in Fig. 2), during bending, at each point of the bar the reaction forces of the foundation are always directed to the normal to the curved bar axis and depend linearly on the bar curvature w(s) at this point. However, changing over to the vertical direction, i.e., parallel to the Oy axis, we obtain a nonlinear dependence $cw(1 - w_s^2)^{1/2}$ of the response of the foundation on the bar deflection (friction between the bar and the foundation is neglected). To accuracy up to fourth-order terms in the function w(s) and its derivatives, the Euler equation for the functional (3.1) is written as

$$EIw_{ssss} + EI(w_{ss}^2 + 4w_s w_{sss})w_{ss} + P(1 + w_s^2/2)w_{ss} + cw(1 - w_s^2/2) = 0. \tag{3.2}$$

Equation (3.2) differs from Eq. (2.3) only in the coefficient of the term cww_s^2. Repeating the operations from item 2, we obtain the prime eigenvalue for $r \neq n^2(n+1)^2$

$$\Lambda_n = n^2 + r/n^2 + \varepsilon\gamma_n^2(n^4 - 2r)/8, \tag{3.3}$$

and its corresponding eigenfunction to accuracy up to ε:

$$W_n(z) = \gamma_n \sin nz - \varepsilon 3(9n^4 - 2r)\gamma_{3n} \sin 3nz/(9n^4 - r)/32\pi.$$

Recall that $\varepsilon = \pi^2(\alpha/L)^2$; hence the value of α corresponding to the buckling mode $w_n = \alpha W_n$ is calculated for a specific value of the load P from the expression (3.3) and is denoted henceforth by α_n. It follows from Eq. (3.3) that the postbuckling behavior of the system with respect to the w_n mode is stable if $n^4 > 2r$, and unstable if $n^4 < 2r$. The points with abscissas $r = n^4/2$ at the straight lines $\Lambda_n^{(0)}$ are denoted by the symbols H_n in Fig. 3. Apparently the point H_n is located to the right of the point H_n^T at the straight line $\Lambda_n^{(0)}$ for any n.

In the case of multiple eigenvalues, i.e., when $r = n^2(n + 1)^2$, apart from the modes w_n and w_{n+1}, in this model the system may take the forms (2.12) for every $n \geq 1$, where ρ_{mj} are connected as before by the relationship (2.10), but $R(n) = n^2(n + 1)^2$. The eigennumbers are expressed by means of the formula (2.13), but with a new value of D(n):

$$D(n) = -[2n^4 - n^2(n + 1)^2 + 2(n + 1)^4]/(n + 1)^2,$$

and the behavior of the system in the form (2.12) is also unstable for every $n \geq 1$, because $D(n) < 0$. As a result, we have the following: In the vicinity of the points K_{12} or K_{23} the postbuckling behavior of the system is stable with respect to the w_2 or w_3 modes, while it is unstable with respect to all other modes. At all other points $K_{n,\,n+1}$ ($n \geq 3$) the postbuckling behavior of the system is unstable with respect to all of the possible curved modes.

Consider in more detail the unstable postbuckling behavior of the system at the points $K_{n,\,n+1}$ ($n \geq 3$). Using the formula (3.3), let us compare the maximum deflections of the modes w_n and w_{n+1} for a fixed value of the load P. Since $(\alpha_{n+1}\gamma_{n+1})^2 = (n^4 - 2r)/[(n + 1)^4 - 2r](\alpha_n\gamma_n)^2 > (\alpha_n\gamma_n)^2$ for any $n \geq 3$, the maximum deflection of the w_{n+1} mode is larger than the deflection of the w_n mode. To compare the maximum deflections of the w_{n+1} and w_n^* modes, we express $\alpha\rho_{nn}$ in terms of α_{n+1} using Eqs. (3.3) and (2.11) with the appropriate value of D_n:

$$(\alpha\rho_{nn})^2 = [(n + 1)^2 - 2n^2]/D(n)\alpha_{n+1}^2 = D_1(n)\alpha_{n+1}^2.$$

Since for any $n \geq 12$

$$\max_{0<z<\pi} w_n^{*2} > D_1(n)(1 + n/(n + 1))^2(\alpha_{n+1}\gamma_{n+1})^2 > (\alpha_{n+1}\gamma_{n+1})^2,$$

upon direct calculations we obtain: 1) the maximum deflection of the w_n^* mode is greater than the deflection of the w_{n+1} mode for every $n \geq 10$, and for $3 \leq n \leq 9$ the deflection of the w_n^* mode is less than the maximum deflection of the w_{n+1} mode, 2) for every n ($1 \leq n \leq 9$) the maximum w_n^* deflection is larger than the deflection of the w_n mode.

Now let us compare the values of potential energies for different buckling modes for a fixed value of the load P. For the w_n mode obtain

$$U_n = L(\pi/L)^6 EI\alpha_n^4\gamma_n^4 n^2[-n^4 - 4r]/64,$$

and for w_{n+1} mode we have

$$U_{n+1} = L(\pi/L)^6 EI\alpha_{n+1}^4\gamma_{n+1}^4(n + 1)^2[-(n + 1)^4 - 4r]/64.$$

For the w_n^* and w_{n+1}^* modes we obtain, accordingly,

$$U_{n,n+1}^* = L(\pi/L)^6 EI\alpha^4\rho_{nn}^4\gamma_n^4 n^4\{[n^2 + (n + 1)^2]R_1(n) \pm 64n^2R_2(n)\}/64, \tag{3.4}$$

where

$$R_1(n) = -\frac{4n^4 + 9n^2(n + 1)^2 + 4(n + 1)^4}{(n + 1)^4};$$

$$R_2(n) = -n\frac{9n^4 - 2n^2(n + 1)^2 + 9(n + 1)^4}{[9(n + 1)^2 - n^2][9n^2 - (n + 1)^2](n + 1)}.$$

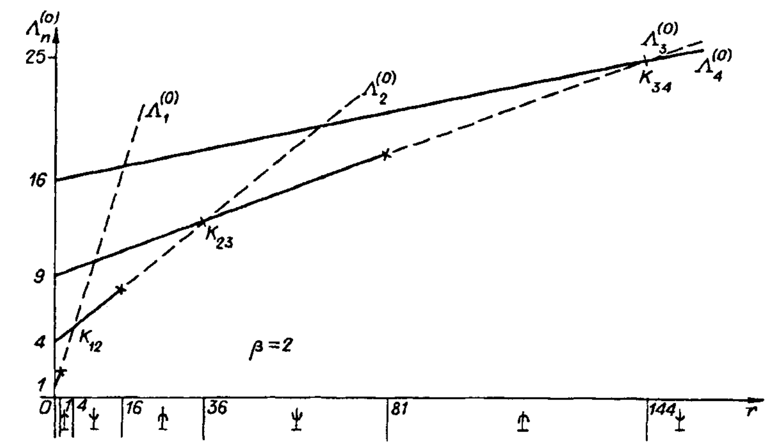

Fig. 6

After simple algebraic operations and direct computations, we obtain a chain of inequalities $0 > U_{n+1}^* > U_n > U_n^*$ for n = 1, 2, $0 > U_{n+1}^* > U_n > U_n^* > U_{n+1}$ for $3 \le n \le 52$, and $0 > U_{n+1}^* > U_n > U_{n+1} > U_n^*$ for $n \ge 53$. Consequently, if we assume that the system selects the state with the least energy, then near the points K_{12}, K_{23}, and $K_{n,\,n+1}$ ($n \ge 53$) (the value of r is fixed, while P changes) the system selects the buckling mode w_n^* among all of the unstable branches. At the other points $K_{n,\,n+1}$ ($3 \le n \le 52$) the system selects the buckling mode w_{n+1}.

4. The Quasi-Winkler Foundation Model. Using the known linear equation [1] of bar equilibrium on an elastic foundation, one can calculate the critical buckling loads in the same manner as for a bar without elastic foundation:

$$EIy_{xxxx} + Py_{xx} + cy = 0. \tag{4.1}$$

However, the postbuckling behavior of the system will remain unexplained. Therefore, consider the following nonlinear equation as a mathematical model describing the buckling behavior of the bar–foundation system

$$EI\varkappa_{xx} + Py_{xx} + cy = 0, \tag{4.2}$$

with curvature

$$\varkappa = y_{xx}/(1 + y_x^2)^{3/2}$$

and boundary conditions

$$y(0) = y(l) = y_{xx}(0) = y_{xx}(l) = 0. \tag{4.3}$$

Using (Fig. 1) the relationships

$$ds = \sqrt{1 + y_x^2}\,dx,\ w(s(x)) = y(x),$$

as well as the relationships

$$y_x = w_s/\sqrt{1 - w_s^2},\ y_{xxx} \approx w_{sss}(1 + 5w_s^2/2) + 4w_s w_{ss}^2,$$
$$y_{xx} = w_{ss}/(1 - w_s^2)^2,\ y_{xxxx} \approx w_{ssss}(1 + 3w_s^2) + 4w_{ss}^2 + 13w_s w_{ss} w_{sss},$$

we write Eq. (4.2) and boundary conditions (4.3) in terms of the function w(s) and variable s. To accuracy up to fourth-order terms in the function w(s) and its derivatives, the initial equation and the appropriate boundary conditions take the form

$$EIw_{ssss} + EI(w_{ss}^2 + 4w_s w_{sss})w_{ss} + P(1 + w_s^2/2)w_{ss} + cw(1 - 3w_s^2/2) = 0,$$
$$w(0) = w(L) = w_{ss}(0) = w_{ss}(L) = 0. \quad (4.4)$$

The equation of the problem (4.4) differs from (2.3) and (3.2) only in the coefficient of the term cww_s^2. Repeating the operations of Paragraphs 2 and 3, we obtain the following results: The postbuckling behavior of the system is stable with respect to the w_n mode when $n^4 > 4r$, and unstable when $n^4 < 4r$. The points with abscissas $r = n^4/4$ are denoted by the symbol H_n^V in Fig. 3. It is evident that the point H_n^V is located to the left of the point H_n^T on the straight line $\Lambda_n^{(0)}$ for every n. Only two modes are possible at the point K_{12}: w_1 and w_2, the postbuckling behavior of the system being unstable with respect to the w_1 mode and neutral to accuracy up to α^2 with respect to the w_2 mode. At the points $k_{n,\,n+1}$ ($n \geq 2$), except for the modes w_n and w_{n+1}, the system may take the form (2.12), where ρ_{mj} are connected by the relationship (2.10), but

$$R(n) = [5n^2 - 2(n+1)^2]/[5(n+1)^2 - 2n^2].$$

The eigennumbers are expressed by means of (2.13), while for this model

$$D(n) = -[12n^4 + 3n^2(n+1)^2 + 12(n+1)^4]/[5(n+1)^2 - 2n^2].$$

At all points $K_{n,\,n+1}$ ($n \geq 2$) the postbuckling behavior of the system is unstable for all buckling modes w_n, w_{n+1}, w_n^*, w_{n+1}^* (Fig. 5). Note that if one takes the expression of potential energy in the form

$$U = \frac{1}{2}EI\int_0^L \varkappa^2 ds - P(L - l) + c\int_0^L\int_0^w w(1 - w_s^2)^{-1/2} dw ds, \quad (4.5)$$

then to accuracy up to fourth-order terms inclusive the equation of the problem (4.4) will be the Euler equation for this functional. The potential energy (4.5) for the modes w_n and w_{n+1} takes the form

$$U_n = L(\pi/L)^6 EI\alpha_n^4 \gamma_n^4 n^2[-n^4 + 10r]/64,$$
$$U_{n+1} = L(\pi/L)^6 EI\alpha_{n+1}^4 \gamma_{n+1}^4 (n+1)^2[-(n+1)^4 + 10r]/64,$$

and for the modes w_n^* and w_{n+1}^* is given by relationship (3.4), where

$$R_1(n) = \frac{46n^4 + 169n^2(n+1)^2 + 46(n+1)^4}{[5(n+1)^2 - 2n^2]^2},$$
$$R_2(n) = \sqrt{\frac{5n^2 - 2(n+1)^2}{5(n+1)^2 - 2n^2}}\,\frac{(n+1)^2[47n^4 - 46n^2(n+1)^2 + 47(n+1)^4]}{[9n^2 - (n+1)^2][9(n+1)^2 - n^2][5(n+1)^2 - 2n^2]}.$$

After algebraic operations and direction calculations, we obtain a chain of inequalities $U_{n+1} > U_n^* > U_n > U_{n+1}^* > 0$ for $2 \leq n \leq 21$ and $U_{n+1} > U_n > U_n^* > U_{n+1}^* > 0$ for $n \geq 22$.

As to the physical sense of the model considered, the following should be noted. If $(1 - w_s^2)^{1/2} = \cos\theta(s)$, one may set in formula (4.5)

$$\int_0^L\int_0^w cw(1 - w_s^2)^{-1/2} dw ds = \int_0^L\int_0^w (cw/\cos\theta(s)) dw ds.$$

In this case the reaction force of an elastic foundation at an arbitrary point s is always directed upright and its projection onto a plane normal to the bar axis is equal in absolute magnitude to cw. On the other hand, since, to accuracy up to fourth-order inclusive in the function w(s) and its derivatives, we have

$$\int_0^L\int_0^w cw(1 - w_s^2)^{-1/2} dw ds \approx \int_0^L\int_0^w cw(1 + w_s^2/2) dw ds =$$
$$\int_0^L\int_0^w [2cw - cw(1 - w_s^2/2)] dw ds \approx c\int_0^L w^2 ds - c\int_0^L\int_0^w w(1 - w_s^2)^{1/2} dw ds,$$

the expression (4.5) can be obtained by subtracting (3.1) from doubled (2.1). Hence, in this case the response of the elastic foundation, up to the given accuracy, can be considered to be equal to the above combination similar responses that were considered in Paragraphs 2 and 3.

5. Generalized Model. Let us consider the family of potential functions

$$U^{\beta} = \frac{1}{2} EI \int_0^L \varkappa^2 ds - P(L - l) + \int_0^L \int_0^w cw(1 - w_s^2)^{\beta/2} dw ds. \tag{5.1}$$

It is evident that the expression (4.5) as well as Eqs. (2.1) and (3.1) taken as starting ones for the models that were concerned in the preceding paragraphs are the particular cases of the expression (5.1) when $\beta = -1$, 0, and 1, respectively. Thus, the formulas given below with the properly chosen values of β coincide with the appropriate formulas of the models considered. The Euler equation corresponding to the functional (5.1) that is dependent on the parameter β can be written, accuracy up to fourth-order terms in the function w(s) and its derivatives, as follows:

$$EIw_{ssss} + EI(w_{ss}^2 + 4w_s w_{sss})w_{ss} + P(1 + w_s^2/2)w_{ss} + cw[1 - (1 - \beta/2)w_s^2] = 0.$$

For the prime eigenvalue the first correction is

$$\Lambda_n^{(1)} = \gamma_n^2 [n^4 - r(3 - \beta)]/8, \tag{5.2}$$

and for the multiple eigenvalue, i.e., when $r = n^2(n + 1)^2$,

$$\Lambda_m^{(1)} = \frac{1}{8}\gamma_n^2 n^2 \frac{(7 - 5\beta)n^4 + (3\beta^2 - 2\beta - 2)n^2(n + 1)^2 + (7 - 5\beta)(n + 1)^4}{(1 - \beta)n^2 - (3 - 2\beta)(n + 1)^2} \rho_{mn}^2.$$

The complex modes of the system deflection are expressed by the formula (2.12), where the coefficients ρ_{mj} are related by the equation

$$\rho_{m,n+1}^2 = \frac{(1 - \beta)(n + 1)^2 - (3 - 2\beta)n^2}{(1 - \beta)n^2 - (3 - 2\beta)(n + 1)^2} \rho_{mn}^2, \tag{5.3}$$

and exist, when $r = n^2(n + 1)^2$, for such n and β for which the coefficient of ρ_{mj} in Eq. (5.3) is greater than zero. For a system that assumes the buckling mode w_n, the potential energy function is expressed as

$$U_n^{\beta} = L(\pi/L)^6 EI\alpha_n^4 \gamma_n^4 n^2 [-n^4 + r(3 - 7\beta)]/64, \tag{5.4}$$

and for the modes w_m, $m = n, n + 1$, we obtain

$$U_m^{\beta} = L(\pi/L)^6 EI\alpha^4 \rho_{mn}^4 \gamma_n^4 n^4 \{[n^2 + (n + 1)^2]R_1 \pm 64n^2(n + 1)^2 \beta R_2\}/64,$$

where

$$R_1 = \frac{(4\beta^3 + 7\beta^2 - 29\beta + 14)[n^4 + (n + 1)^4] - (31\beta^3 - 84\beta^2 + 58\beta + 4)n^2(n + 1)^2}{[(1 - \beta)n^2 - (3 - 2\beta)(n + 1)^2]^2};$$

$$R_2 = \sqrt{\frac{(1 - \beta)(n + 1)^2 - (3 - 2\beta)n^2}{(1 - \beta)n^2 - (3 - 2\beta)(n + 1)^2}} \frac{(28 - 19\beta)[n^4 + (n + 1)^4] - (22\beta - 24)n^2(n + 1)^2}{[9(n + 1)^2 - n^2][9n^2 - (n + 1)^2][(1 - \beta)n^2 - (3 - 2\beta)(n + 1)^2]}.$$

Now, let us analyze the expression (5.4), which takes the form of a germ of the dual cusp catastrophe when $r = 0$. In order for the expression with $r \geq 0$ (5.4) to take the above form, it is necessary and sufficient for the inequality $-n^4 + r(3 - 7\beta) < 0$ to be met for all r from the interval $(n - 1)^2 n^2 \leq r \leq n^2(n + 1)^2$, when the postbuckling behavior of the system in the vicinity of the first critical load is concerned, i.e., provided that $\beta > 11/28$. If $\beta > 3/7$, then the inequality $-n^4 + r(3 - 7\beta) < 0$ is fulfilled for any $r \geq 0$, i.e., the expression U_n^{β} of the potential energy function of the bar–foundation system which assumes any buckling mode w_n is always in the form of a germ of the dual cusp catastrophe when $\beta > 3/7$. There is no violation of the structural stability requirements [11] for such values of β. Recall that, of the three models considered, the inequality $\beta > 3/7$ is satisfied solely by the model in Paragraph 3 for which $\beta = 1$. On the above basis it may be concluded that this model describes the postbuckling behavior of the bar–foundation system most adequately. For the model, the potential energy function expressions are in the form of a germ of the dual cusp catastrophe for every buckling mode and value of n and r, i.e., there is no violation of the structural stability requirement for a family of the potential functions.

Analysis of Eq. (5.3) shows that, if $1/2 < \beta < 11/7$, then apart from w_n and w_{n+1} for the dimensionless foundation stiffness coefficient $r = n^2(n + 1)^2$, where n is any natural number, complex buckling modes $w_m{}^*$ are possible. There is no w_m mode, and when $\beta = 2$, complex modes are possible for $r = n^2(n + 1)^2$ for the other β, when n is larger than some $N > 1$. As can be seen from Eq. (5.2), when $\beta \geq 3$, for any $r > 0$, the postbuckling behavior with respect to each w_n mode is stable; however, for $\beta < 3$, the behavior becomes unstable, when $r > n^4/(3-\beta)$. Moreover, if $\beta \leq 2$, then for every w_n mode there is always a value of r^* of the interval $(n-1)^2n^2 \leq r^* < n^2(n + 1)^2$ such that the postbuckling behavior of the system is unstable with respect to the mode w_n, where $r \geq r^*$. Note that it suffices to take any r from the interval $n^4 < r < n^2(n + 1)^2$ as such r^*.

Figure 6 shows a plot of the dimensionless parameter $\Lambda_n{}^{(0)}$ of the critical load versus the dimensionless stiffness coefficient of the foundation r for a model where $\beta = 2$. The line segments $\Lambda_n{}^{(0)}$ close to which the postbuckling behavior of the system is stable are shown in Fig. 6 by the solid line; and close to which it is unstable, by the broken line. For $\beta < 2$ the value of r at which the postbuckling behavior of the system with respect to the mode w_n becomes neutral is less than that when $\beta = 2$ (in Fig. 6 the corresponding points on the straight lines are labeled by crosses). In the models with $\beta > 11/4$ only, the postbuckling behavior of the system near the first critical load is stable for every r. However, when the second correction of $\Lambda_n{}^{(2)}$ for the eigenvalue $\Lambda_n{}^{(0)}$ of the linearized problem is taken into account, the situation changes for the worse or, more precisely, the domain of stable postbuckling behavior of the system is narrowed. In fact, the second correction is

$$\Lambda_n^{(2)} = \frac{21}{512}\gamma_n^4 n^6 + \frac{\gamma_n^4 n^2}{512} r\left[-24\beta^2 + 26\beta - 33 + \frac{(\beta - 2)(5\beta + 6)}{9n^4/r - 1}\right]. \tag{5.5}$$

Let us show that $\Lambda_n{}^{(2)} < 0$ for every β and $n^4 < r < n^2(n + 1)^2$. For these values of ρ the inequality $9n^4/r - 1 > 5/4$ is valid. Therefore, for $\Lambda_n{}^{(2)}$ we have the estimate

$$\Lambda_n^{(2)} < \gamma_n^4 n^2\{21n^4 + r[-24\beta^2 + 26\beta - 33 + 4(\beta - 2)(5\beta + 6)/5]\}/512 =$$
$$\gamma_n^4 n^2\{21n^4 - r(100\beta^2 - 114\beta + 213)/5\}/512 < \gamma_n^4 n^2(21n^4 - 36r)/512.$$

Hence, $\Lambda_n{}^{(2)} < 0$ for $r > 7n^4/12$, i.e., everywhere where $\Lambda_n{}^{(1)} < 0$.

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